## Gradient, Divergence and curl

First we define the vector differential operator $\vec{\nabla}$. It is defined as

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$
$$\equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

**Gradient:** Suppose $\phi$ is a scalar function of $(x,y,z)$ and $\phi$ is differentiable at each $(x,y,z)$. Then gradient of $\phi$ is denoted by $\vec{\nabla}\phi$ is defined as

$$\cancel{\nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \hat{k}\frac{\partial\phi}{\partial z}}$$
$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$
$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

so $\vec{\nabla}\phi$ is a vector field.

**Def:** suppose $\hat{a}$ is a unit vector, then the component of $\vec{\nabla}\phi$ along $\hat{a}$ is denoted by $\vec{\nabla}\phi\cdot\hat{a}$ and it is called directional derivative of $\phi$ along $\hat{a}$ or in the direction of $\hat{a}$.

Ex: Properties: If $\varphi, \varphi'$ are two scalar fields ~~these~~ which are differentiable at each $(x,y,z)$.

1) $\vec{\nabla}(\varphi+\varphi') = \vec{\nabla}\varphi + \vec{\nabla}\varphi'$

2) $\vec{\nabla}(\varphi\varphi') = \varphi\vec{\nabla}\varphi' + \varphi'\vec{\nabla}\varphi$

1) H.W

② 

$$\vec{\nabla}(\varphi\varphi') = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\varphi\varphi')$$

$$\cancel{= \varphi'\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\varphi}$$

$$\cancel{+ \varphi\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\right)\varphi'}$$

$$= \varphi'\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\varphi + \varphi\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\varphi'$$

$$= \varphi'\left(\hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z}\right) + \varphi\left(\hat{i}\frac{\partial\varphi'}{\partial x} + \hat{j}\frac{\partial\varphi'}{\partial y} + \hat{k}\frac{\partial\varphi'}{\partial z}\right)$$

$$= \varphi'\vec{\nabla}\varphi + \varphi\vec{\nabla}\varphi'$$

Ex: Suppose $\varphi = \frac{1}{r}$.

where $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

find $\vec{\nabla}\varphi$.

Sol: $\phi = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-\frac{1}{2}}$

$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}}$

$+ \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}$

$= \hat{i}\left(-\frac{1}{2}\right)\frac{2x}{(x^2+y^2+z^2)^{3/2}} + \hat{j}\left(-\frac{1}{2}\right)\frac{2y}{(x^2+y^2+z^2)^{3/2}}$

$+ \hat{k}\left(-\frac{1}{2}\right)\frac{2z}{(x^2+y^2+z^2)^{3/2}}$

$= -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\vec{r}}{r^3}$

① Let $\phi = 2xz^4 - x^2y$ find $\vec{\nabla}\phi$ and $|\vec{\nabla}\phi|$ at $(2,-2,-1)$

Sol: $\phi = 2xz^4 - x^2y$

$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$= (2z^4 - 2xy)\hat{i} + (-x^2)\hat{j} + (8xz^3)\hat{k}$

$\vec{\nabla}\phi|_{(2,-2,-1)} = (2+8)\hat{i} + (-4)\hat{j} + (-16)\hat{k}$

$= 10\hat{i} - 4\hat{j} - 16\hat{k}$

So $|\vec{\nabla}\phi|$

$= \sqrt{(2z^4-2xy)^2 + (-x^2)^2 + (8xz^3)^2}$

$|\nabla\phi|_{(2,-2,-1)} = 2\sqrt{93}$

Scanned with CamScanner

2. $\phi(x,y,z) = 3x^2y - y^3z^2$

find $\vec{\nabla}\phi$ at $(1,-2,-1)$